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Abstract. We consider series defined by means of the two-, three- and four-index generalized Bessel functions and study the behaviour of such series on the boundaries of their convergence domains. Analogues of the classical theorems for the power series like overconvergence, as well as Hadamard type theorems are proposed.

1. INTRODUCTION

Let J_ν and C_ν denote respectively the classical Bessel function

$$J_\nu(z) = (z/2)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{k!\Gamma(k+\nu+1)}, \quad \nu \in \mathbb{C},\ z \in \mathbb{C} \setminus (-\infty, 0], \tag{1.1}$$

and Bessel-Clifford function $C_\nu(z) = z^{-\nu/2} J_\nu(2\sqrt{z}) \quad (\nu \in \mathbb{C},\ z \in \mathbb{C})$.

The generalization of the Bessel-Clifford function

$$J_\nu^\mu(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\ \Gamma(\nu + \mu k + 1)} \qquad (\mu > -1,\ z \in \mathbb{C}), \tag{1.2}$$

was introduced by Wright and called Bessel-Wright or misnamed as Bessel-Maitland function (after Sir Edward Maitland Wright). Initially, Wright defined (1.2) only for $\mu > 0$, and on a later stage extended its definition to $\mu > -1$ (see for example [1, 2]).

More general are the three- and four-index generalizations of the Bessel function J_ν, namely generalized Bessel-Maitland function:

$$J_{\nu,\lambda}^\mu(z) = (z/2)^{\nu+2\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{\Gamma(\lambda+k+1)\Gamma(\nu+k\mu+\lambda+1)} \qquad (\mu > 0,\ \nu, \lambda \in \mathbb{C},\ z \in \mathbb{C} \setminus (-\infty, 0]), \tag{1.3}$$

introduced by Pathak (for details see [1, 2]), and the generalized Lommel-Wright function with 4 indices ($\mu > 0, q \in \mathbb{N}, \nu, \lambda \in \mathbb{C}$), introduced by de Oteiza, Kalla and Conde (for details see [1, 2]):

$$J_{\nu,\lambda}^{\mu,q}(z) = (z/2)^{\nu+2\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{(\Gamma(\lambda+k+1))^q\, \Gamma(\nu+k\mu+\lambda+1)} \qquad (\mu > 0,\ q \in \mathbb{N},\ \nu, \lambda, \in \mathbb{C},\ z \in \mathbb{C} \setminus (-\infty, 0])\cdot \tag{1.4}$$

Now, consider the families of Bessel-Maitland functions with integer indices ν, i.e. the Bessel-Maitland functions

$$J_n^\mu(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\ \Gamma(n+\mu k+1)}, \quad n = 0, 1, 2, \ldots \tag{1.5}$$

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and the functions (1.3), (1.4) for indices of the kind $\nu = n - 2\lambda$; $n = 0, 1, 2, ...$, i.e. the generalized Bessel-Maitland function with 3 indices

$$J^{\mu}_{n-2\lambda,\lambda}(z) = (z/2)^n \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{\Gamma(\lambda+k+1)\Gamma(n-\lambda+k\mu+1)}, \quad n = 0, 1, 2, \ldots \tag{1.6}$$

and the generalized Lommel-Wright function with 4 indices

$$J^{\mu,q}_{n-2\lambda,\lambda}(z) = (z/2)^n \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{(\Gamma(\lambda+k+1))^q\,\Gamma(n-\lambda+k\mu+1)}, \quad n = 0, 1, 2, \ldots \tag{1.7}$$

In this paper we study series in such a kind of entire functions and their convergence and overconvergence.

2. ASYMPTOTIC FORMULAE

The asymptotic formula

$$J^{\mu}_n(z) = \frac{1}{\Gamma(n+1)}(1 + \theta^{\mu}_n(z)), \quad z \in \mathbb{C},\ \mu > 0; \quad \theta^{\mu}_n(z) \to 0 \ \text{ as } \ n \to \infty \tag{2.1}$$

with respect to the index n is given in [3] for the Bessel-Maitland functions. The functions $\theta^{\mu}_n(z)$ are entire functions. The convergence of $\left\{\theta^{\mu}_n(z)\right\}$ is uniform on the compact subsets of the complex plane $\mathbb{C}$. Considering explicitly $\theta^{\mu}_n(z)$, this result can be made sharper, as follows.

Lemma 1. *Let $K \subset \mathbb{C}$ be a nonempty compact set. Then there exists a constant $C = C(K)$, $0 < C < \infty$, such that for each $n \in \mathbb{N}_0$ and each $z \in K$ the following inequality holds*

$$|\theta^{\mu}_n(z)| \le C\,\Gamma(n+1)/\Gamma(n+1+\mu). \tag{2.2}$$

Further, Stirling's formula (see e.g. [4]) gives that

$$\frac{\Gamma(n+1)}{\Gamma(n+1+\mu)} = O\left(\frac{1}{n^{\mu}}\right), \quad \text{for } n \in \mathbb{N}. \tag{2.3}$$

Consider now the generalized Lommel-Wright function with 4 indices for indices of the kind $\nu = n - 2\lambda$; $n = 0, 1, 2, ...$ Given a number λ, suppose that some coefficients in (1.7) equal zero, that is, there exist numbers $p \in \mathbb{N}_0$ and $s \in \mathbb{N}$, such that the identity (1.7) can be written as

$$J^{\mu,q}_{n-2\lambda,\lambda}(z) = (z/2)^n \left(\frac{(-1)^p (z/2)^{2p}}{(\Gamma(\lambda+p+1))^q\,\Gamma(n-\lambda+p\mu+1)} + \sum_{k=p+s}^{\infty} \frac{(-1)^k (z/2)^{2k}}{(\Gamma(\lambda+k+1))^q\,\Gamma(n-\lambda+k\mu+1)} \right). \tag{2.4}$$

As it is well known the function $J^{\mu,q}_{n-2\lambda,\lambda}$ is an entire function (see e.g. [5]). The corresponding asymptotic result one can write in the following way (see for details [6]).

Lemma 2. *Let $\mu > 0$. Then the generalized Lommel-Wright functions* (1.7) *satisfy the following asymptotic formula*

$$J^{\mu,q}_{n-2\lambda,\lambda}(z) = \frac{(-1)^p (z/2)^{n+2p}}{(\Gamma(\lambda+p+1))^q\,\Gamma(n-\lambda+p\mu+1)}(1 + \theta^{\mu,q}_{n-2\lambda,\lambda}(z)), \quad \theta^{\mu,q}_{n-2\lambda,\lambda}(z) \to 0 \ \text{ as } \ n \to \infty. \tag{2.5}$$

On the compact subsets of the complex plane $\mathbb{C}$ the convergence is uniform and

$$\theta^{\mu,q}_{n-2\lambda,\lambda}(z) = O\left(\frac{1}{n^{s\mu}}\right), \quad (n \in \mathbb{N}) \tag{2.6}$$

with the corresponding $s \in \mathbb{N}$, depending on λ.

Remark 3. *By the way, having in view that the functions* (1.6) *can be obtained from* (1.7) *setting $q = 1$, the asymptotic formula for the generalized Bessel-Maitland functions follows just as a particular case of the above given.*

Remark 4. *The uniform convergence of $\theta^{\mu}_n(z)$, $\theta^{\mu}_{n-2\lambda,\lambda}$ and $\theta^{\mu,q}_{n-2\lambda,\lambda}$ on the compact subsets of $\mathbb{C}$ follows from Lemmas 1 and 2 and formulae* (2.3), (2.6)*, as well.*

3. SERIES IN GENERALIZED BESSEL FUNCTIONS

Multiplying the functions (1.5)–(1.7) with suitable coefficients and power functions, and taking into account the relation (2.4), we obtain a little bit modified enumerable systems of functions as follows:

$$\widetilde{J}_n^{\mu}(z) = z^n \Gamma(n+1) J_n^{\mu}(z), \quad n = 0, 1, 2, \ldots \tag{3.1}$$

$$\widetilde{J}_{n-2\lambda,\,\lambda}^{\mu}(z) = 2^{n+2p}\,\Gamma(\lambda+p+1)\Gamma(n-\lambda+p\mu+1) z^{-2p} J_{n-2\lambda,\,\lambda}^{\mu}(z), \quad n = 0, 1, 2, \ldots \tag{3.2}$$

$$\widetilde{J}_{n-2\lambda,\,\lambda}^{\mu,\,q}(z) = 2^{n+2p}(\Gamma(\lambda+p+1))^q\,\Gamma(n-\lambda+p\mu+1) z^{-2p} J_{n-2\lambda,\,\lambda}^{\mu,\,q}(z), \quad n = 0, 1, 2, \ldots \tag{3.3}$$

For convenience, supposing $\mu > 0$, let us briefly denote by $\widetilde{J}$ any of the families listed above, i. e. ($\widetilde{j}_n$ are the functions, belonging to the corresponding family):

$$\widetilde{J} := \left\{\widetilde{j}_n\right\}_{n\in\mathbb{N}_0}.$$

Observe that $\widetilde{j}_n$ can be written in the form

$$\widetilde{j}_n(z) = z^n(1+\widetilde{\theta}_n(z)), \quad \widetilde{\theta}_n(z) \to 0 \text{ as } n \to \infty, \tag{3.4}$$

with the respective $\widetilde{\theta}_n$ and uniform convergence on the compact subsets of $\mathbb{C}$. Moreover,

$$\widetilde{\theta}_n(z) = O\left(\frac{1}{n^{s\mu}}\right), \quad s \in \mathbb{N}, \tag{3.5}$$

($s = 1$ in the case (3.1)), confer with (2.1) – (2.3), (2.5), (2.6).

Remark 5. *According to the asymptotic formulae* (2.1) *and* (2.5) *and* (3.4), (3.5) *as well, it follows that there exists a natural number N_0 such that the functions $\widetilde{j}_n$ have no zeros for $n > N_0$, except for the origin.*

Remark 6. *Note that each function $\widetilde{j}_n$ ($n \in \mathbb{N}_0$), being an entire function, no identically zero, has no more than finite number of zeros in the closed and bounded set $|z| \le R$ (see [7], vol.1, ch. 3, §6, 6.1, p. 305). Moreover, because of Remark 5, no more than finite number of these functions have some zeros, except for the origin.*

We consider the series in discussed above generalized Bessel functions in the complex plane and we briefly call them *Bessel type series*. Namely, we consider the series

$$\sum_{n=0}^{\infty} a_n \widetilde{J}_n^{\mu}(z), \quad \sum_{n=0}^{\infty} a_n \widetilde{J}_{n-2\lambda,\,\lambda}^{\mu}(z), \quad \sum_{n=0}^{\infty} a_n \widetilde{J}_{n-2\lambda,\,\lambda}^{\mu,\,q}(z) \qquad (\mu > 0; \quad a_n,\, z \in \mathbb{C}), \tag{3.6}$$

or, which is the same, the series

$$\sum_{n=0}^{\infty} a_n \widetilde{j}_n(z), \quad a_n \in \mathbb{C}, \quad z \in \mathbb{C}. \tag{3.7}$$

Further, we use the introduced denotation (3.7), understanding each series in (3.6).

4. CAUCHY-HADAMARD, ABEL AND FATOU TYPE THEOREMS

In the beginning, we state a theorem of Cauchy-Hadamard type (proven in [8, 9]) and a corollary for the discussed series, giving their domains of convergence.

In what follows we use the notation $D(0;R)$ and $C(0;R)$ respectively for the open disk centered at the origin with a radius R and its boundary, i.e.:

$$D(0;R) = \{z : |z| < R,\ z \in \mathbb{C}\}, \quad C(0;R) = \partial D(0;R) = \{z : |z| = R,\ z \in \mathbb{C}\}.$$

Theorem 7 (of Cauchy-Hadamard type)**.** *The region of convergence of the series* (3.7) *with complex coefficients* a_n *is the disk* $D(0;R)$ *with a radius of convergence*

$$R = \left(\limsup_{n\to\infty} (\ |a_n|\ )^{1/n}\right)^{-1}. \tag{4.1}$$

More precisely, the series is absolutely convergent in the disk D(0;R) and divergent in the region $|z| > R$*. The cases* $R = 0$ *and* $R = \infty$ *fall in the general case.*

So, the considered series (3.7) converges in a respective disk, like in the theory of the widely used power series. Analogously, inside the corresponding disk, the convergence of the discussed series is uniform, i.e., the following corollary, similar to the classical Abel lema, holds.

Corollary 8. *Let the series* (3.7) *converges at the point* $z_0 \neq 0$*. Then it is absolutely convergent in the disk* $D(0;|z_0|)$*. Inside the disk* $D(0;R)$*, i.e. on each closed disk* $|z| \leq r < R$ *(*R *defined by* (3.1)*), the convergence is uniform.*

The very disk of convergence is not obligatorily a domain of uniform convergence and on its boundary the series may even be divergent.

Let $z_0 \in \mathbb{C}$, $0 < R < \infty$, $|z_0| = R$ and g_φ be an arbitrary angular domain with size $2\varphi < \pi$ and with a vertex at the point $z = z_0$, which is symmetric with respect to the straight line defined by the points 0 and z_0 and d_φ be the part of the angular domain g_φ, closed between the angle's arms and the arc of the circle with center at the point 0 and touching the arms of the angle.

The following theorem refers to the uniform convergence of the series (3.7) in the set d_φ and the limit of its sum at the point z_0, provided $z \in D(0;R) \cap g_\varphi$.

Theorem 9 (of Abel type)**.** *Let* $\{a_n\}_{n=0}^{\infty}$ *be a sequence of complex numbers, R be the real number defined by* (4.1) *and* $0 < R < \infty$*. If* $\widetilde{f}(z)$ *is the sum of the series* (3.7) *in the domain* $D(0;R)$*, i.e.*

$$\widetilde{f}(z) = \sum_{n=0}^{\infty} a_n \widetilde{j}_n(z), \quad z \in D(0;R),$$

and this series converges at the point z_0 *of the boundary* $C(0;R)$*, then:*

(i) *The following relation holds*

$$\lim_{z\to z_0} \widetilde{f}(z) = \sum_{n=0}^{\infty} a_n \widetilde{j}_n(z_0),$$

provided $z \in D(0;R) \cap g_\varphi$*;*

(ii) *The series* (3.7) *is uniformly convergent in the domain* d_φ*.*

Let us consider the power series $\sum\limits_{n=0}^{\infty} a_n z^n$ with complex coefficients a_n and a radius of convergence $0 < R < \infty$, $f(z)$ be the sum of this series in the disk of convergence $D(0;R)$, i.e.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in D(0;R), \tag{4.2}$$

and $z_0 \in \partial D(0;R)$ be a regular point for f. Then the series (4.2) may be convergent or divergent at z_0. Practically, there is no relation between the convergence (divergence) of a power series at points on the boundary of its disk of convergence and the regularity (singularity) of its sum of such points. For example: The power series $\sum\limits_{n=0}^{\infty} z^n$ is divergent at each point of the unit circle $C(0;1)$ regardless of the fact that all the points of this circle, except for $z = 1$, are regular for its sum. The series $\sum\limits_{n=1}^{\infty} n^{-2} z^n$ is (absolutely) convergent at each point of the circle $C(0;1)$, but nevertheless one of them, namely $z=1$, is a singular (i.e. not regular) for its sum. However, under additional conditions on the sequence $\{a_n\}_{n=0}^{\infty}$, such a relation does exist (see for details Fatou theorem in [7], Vol.1, Ch. 3, §7, 7.3, p.357). Namely, the following theorem holds true.

Theorem 10 (of Fatou)**.** *If the coefficients of the power series with the unit disk of convergence tend to the zero, i.e.* $\lim\limits_{n\to\infty} a_n = 0$*, then the power series converges, even uniformly, on each arc of the unit circle, all the points of which (including the ends of the arc) are regular for the sum of the series.*

Propositions referring to the properties discussed above have been also established for series in other special functions, e.g. in the Laguerre and Hermite polynomials, as well as in Bessel and Mittag-Leffler systems (see [10], resp. [11, 12, 13]). Here we give such a type of theorem for the generalized Bessel systems, proven in [14].

Theorem 11 (of Fatou type)**.** *Let* $\{a_n\}_{n=0}^{\infty}$ *be a sequence of complex numbers satisfying the conditions*

$$\lim_{n\to\infty} a_n = 0, \qquad \limsup_{n\to\infty} (\,|a_n|\,)^{1/n} = 1, \tag{4.3}$$

and $\widetilde{f}(z)$ *be the sum of the series* (3.7) *in the unit disk* $D(0;1)$*, i.e.*

$$\widetilde{f}(z) = \sum_{n=0}^{\infty} a_n \widetilde{j}_n(z), \qquad z \in D(0;1).$$

Let σ *be an arbitrary arc of the unit circle* $C(0;1)$ *with all its points (including the ends) regular to the function* $\widetilde{f}$*. Then the series* (3.6) *converges, even uniformly, on the arc* σ*.*

5. AN OVERCONVERGENCE THEOREM

Recall that it is possible a given power series with a finite radius of convergence $0 < R < \infty$ to be convergent or divergent at some points of the boundary $C(0;R)$. These points could be regular or singular for its sum f, but the series diverges outside the domain of convergence. However, sometimes it is possible a subsequence of its partial sums to exist that converges in a neighbouhood of a regular point of the sum. In order to introduce the next two definitions ([7, Vol. 2, p. 500]) and to expose the results, obtained in this section, we first set

$$s_p(z) = \sum_{k=0}^{p} a_k z^k, \quad S_p(z) = \sum_{k=0}^{p} a_k \widetilde{j}_k(z), \qquad p = 0,1,2,\ldots \tag{5.1}$$

Definition 12. *A power series with a finite radius of convergence R is said to be overconvergent, if there exist a subsequence* $\left\{s_{p_k}\right\}_{k=0}^{\infty}$ *of the partial-sums sequence* $\left\{s_p\right\}_{p=0}^{\infty}$ *and a region G, containing the open disk* $D(0;R)$*,* $G \cap \partial D(0;R) \neq \varnothing$*, such that* $\left\{s_{p_k}\right\}$ *is uniformly convergent inside G.*

Definition 13. *We say that the function f (or the series), given by* (4.2)*, possesses Hadamard gaps, if there exist two sequences* $\{p_n\}_{n=0}^{\infty}$ *and* $\{q_n\}_{n=0}^{\infty}$*, having the property* $q_{n-1} \le p_n \le q_n/(1+\theta)$ $(\theta > 0)$ *and* $a_k = 0$ *for* $p_n < k < q_n$ $(n = 0,1,2,\ldots)$.

Thus, beginning with the domain of convergence and series behaviour near its boundary, passing through the possible uniform convergence on an arbitrary closed arc of the boundary, all the points of which are regular for its sum f, we come to the natural question: "*What type of conditions should be imposed on the power series that ensure the existence of subsequence* $\left\{s_{p_k}\right\}$*, convergent outside the disk of convergence?*". The answer to this question is given in the early 20th century by Ostrowski [15, 16], see also [17]. Namely, his classical result states that a given power series with existing regular points on the boundary of convergence disk is overconvergent iff it possesses Hadamard gaps. We draw the attention to the fact that merely the existence of Hadamard gaps does not imply overconvergence.

For example, the power series $\sum\limits_{n=0}^{\infty} a_{k_n} z^{k_n}$ with $k_{n+1} \geq (1+\theta)k_n$ $(\theta > 0)$ and $\limsup\limits_{n\to\infty} (\, |a_{k_n}|\,)^{1/k_n} = 1$ possesses Hadamard gaps but nevertheless it is not overconvergent. Its natural boundary of analyticity is the unit circle $|z| = 1$ and that is nothing but the theorem about the gaps, belonging to Hadamard [18].

Remark 14. *To introduce the corresponding notions 'overconvergence' and 'gaps' for the series* (3.7)*, the expression z^n has to be replaced by $\widetilde{j}_n(z)$ and, respectively, the sequence $\{s_{p_k}\}$ by the sequence $\{S_{p_k}\}$.*

Theorem 15 (of overconvergence)**.** *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers satisfying the condition $\limsup\limits_{n\to\infty} (\, |a_n|\,)^{1/n} = 1$, $\widetilde{f}(z)$ be the sum of the series* (3.7) *in the unit disk $D(0;1)$, $\widetilde{f}(z)$ have at least one regular point, belonging to the circle $C(0;1)$, and let $\widetilde{f}(z)$ possess Hadamard gaps. Then the series* (3.7) *is overconvergent.*

Proof. Without loss the generality we suppose that the point $z_0 = 1$ is regular to the function $\widetilde{f}$. That means that $\widetilde{f}$ is analytically continuable in a neighbourhood U of the point 1.

Denoting $\widetilde{U} = U \cup D(0;1)$, we define the function ψ in the region $\widetilde{U}$ by the equality

$$\psi(z) = \widetilde{f}(z), \quad z \in D(0;1).$$

More precisely, it means that ψ is a single valued analytical continuation of $\widetilde{f}$ into the domain $\widetilde{U}$.

Letting $\theta > 0$ and taking $\{p_n\}_{n=0}^{\infty}$, $\{q_n\}_{n=0}^{\infty}$ with the properties $q_n \geq (1+\theta)p_n$, $p_n \geq q_{n-1}$ and $a_k = 0$ for $p_n < k < q_n$ $(n = 0, 1, 2, \dots)$, we define the auxiliary function

$$\varphi_n(z) = \psi(z) - S_{p_n} = \psi(z) - \sum_{k=0}^{p_n} a_k \widetilde{j}_k(z). \tag{5.2}$$

In order to prove that the sequence $\left\{S_{p_n}\right\}$ is uniformly convergent inside the region $\widetilde{U}$, we are going to apply the Hadamard theorem for the three disks [7, Vol. 2]. To this end, taking δ and ω in such a way that $0 < \omega < \delta < 1/2$, we consider the three circles C_1, C_2, C_3, centered at the point $1/2$ and having respectively radii $1/2 - \delta$, $1/2 + \omega$, $1/2 + \delta$, such that $C_3 \subset \widetilde{U}$ and after that set

$$M_{n,j} = \max_{z\in C_j} |\varphi_n(z)| \quad j = 1,2,3; \qquad M = \max_{z\in C_3} |\psi(z)|.$$

Before evaluating $|\varphi_n(z)|$ we come back to (3.4) and (3.5). Just mention that since $\lim\limits_{n\to\infty} \dfrac{1}{n^{s\mu}} = 0$, in view of (3.4) and (3.5), there exists a number B such that $|1 + \widetilde{\theta}_n(z)| \leq B$ for all the values of $n \in \mathbb{N}$ on an arbitrary compact subset of $\mathbb{C}$. Now, letting $0 < \eta < \delta/2$ implies the existence of $A = A(\eta)$ such that $|a_k| \leq AB^{-1}(1-\eta)^{-k}$. To find an upper estimation of $|\varphi_n(z)|$ we intend to consider three different cases.

I. First, let $z \in C_1 \subset D(0;1)$. In the unit disk, according to (5.2), we have

$$\varphi_n(z) = \sum_{k=q_n}^{\infty} a_k \widetilde{j}_k(z).$$

Therefore,

$$|\varphi_n(z)| \leq \sum_{k=q_n}^{\infty} |a_n \widetilde{j}_k(z)| = \sum_{k=q_n}^{\infty} |a_k z^k (1+\theta_k(z))| = \sum_{k=q_n}^{\infty} |a_k||1+\theta_k(z)||z^k|$$

$$\leq A \sum_{k=q_n}^{\infty} (1-\eta)^{-k}(1-\delta)^k = A\left(1 - \frac{1-\delta}{1-\eta}\right)^{-1} \left(\frac{1-\delta}{1-\eta}\right)^{q_n},$$

whence

$$M_{n,1} = O\left(\left(\frac{1-\delta}{1-\eta}\right)^{q_n}\right) = O\left(\left(\frac{1-\delta}{1-\eta}\right)^{(1+\theta)p_n}\right). \tag{5.3}$$

II. Now, let $z \in C_3$. In this case

$$|\varphi_n(z)| = |\psi(z) - S_{p_n}| = |\psi(z) - \sum_{k=0}^{p_n} a_k \widetilde{j}_k(z)| \le |\psi(z)| + \sum_{k=0}^{p_n} |a_k \widetilde{j}_k(z)|$$

$$\le M + \sum_{k=0}^{p_n} |a_k||1 + \theta_k(z)||z^k| \le M + A \sum_{k=0}^{p_n} \left(\frac{1+\delta}{1-\eta}\right)^k = O\left(\left(\frac{1+\delta}{1-\eta}\right)^{p_n}\right),$$

and therefore

$$M_{n,3} = O\left(\left(\frac{1+\delta}{1-\eta}\right)^{p_n}\right). \tag{5.4}$$

III. Finally, let $z \in C_2$. Then, in view of (5.3) and (5.4) and according to the Hadamard theorem for the three disks (for details see [7, Vol. 2, formula (3.2:2)]), we can write

$$M_{n,2} = O\left(\left(\left(\frac{1-\delta}{1-\eta}\right)^{(1+\theta)\ln\frac{1+2\delta}{1+2\omega}} \left(\frac{1+\delta}{1-\eta}\right)^{\ln\frac{1+2\omega}{1-2\delta}}\right)^{p_n}\right). \tag{5.5}$$

Note that the limit of the inner part of (5.5), namely

$$\left(\frac{1-\delta}{1-\eta}\right)^{(1+\theta)\ln\frac{1+2\delta}{1+2\omega}} \left(\frac{1+\delta}{1-\eta}\right)^{\ln\frac{1+2\omega}{1-2\delta}},$$

is equal to

$$a = (1-\delta)^{(1+\theta)\ln(1+2\delta)}(1+\delta)^{-\ln(1-2\delta)}, \tag{5.6}$$

when ω and η tend to 0. Additionally, if δ tends to 0 then $a < 1$. Indeed, taking the logarithm of a, we have

$$\ln a = (1+\theta)\ln(1+2\delta)\ln(1-\delta) - \ln(1-2\delta)\ln(1+\delta)$$

$$= (1+\theta)(2\delta + O(\delta^2))(-\delta + O(\delta^2)) - (-2\delta + O(\delta^2))(\delta + O(\delta^2))$$

$$= (1+\theta)(-2\delta^2 + O(\delta^3)) + 2\delta^2 + O(\delta^3) = -2\theta\delta^2 + O(\delta^3).$$

Therefore $\ln a < 0$ when $\delta \to 0$ and for this reason $a < 1$ if δ tends to 0.

That is why, $\lim_{n\to\infty} M_{n,2} = 0$, whence $\left\{S_{p_n}\right\}$ is uniformly convergent inside the region $\widetilde{U}$. □

Theorem 16 (of Hadamard type about the gaps). *Let* $\{a_k\}_{k=0}^{\infty}$ *be a sequence of complex numbers satisfying the conditions* $\limsup_{n\to\infty}(\,|a_{k_n}|\,)^{1/k_n} = 1$, $k_{n+1} \ge (1+\theta)k_n$ $(\theta > 0)$, $a_k = 0$ *for* $k_n < k < k_{n+1}$ *and* $\widetilde{f}(z)$ *be the sum of the series* (3.7) *in the unit disk* $D(0;1)$*, i.e.*

$$\widetilde{f}(z) = \sum_{n=0}^{\infty} a_{k_n} \widetilde{j}_{k_n}(z), \quad z \in D(0;1).$$

Then all the points of the unit circle $C(0;1)$ *are singular for the function* $\widetilde{f}$*, i.e. the unit circle is a natural boundary of analyticity for the series*

$$\sum_{n=0}^{\infty} a_{k_n} \widetilde{j}_{k_n}(z). \tag{5.7}$$

Proof. Indeed, let $|z_0| = 1$ and z_0 be regular for $\widetilde{f}$. Denoting $p_n = k_n$, $q_n = k_{n+1}$, we can conclude that $q_{n-1} \le p_n \le q_n/(1+\theta)$ $(\theta > 0)$ and $S_{p_n} = \sum_{s=0}^{n} a_{k_s} \widetilde{j}_{k_s}(z)$. Therefore, analogously to the proof of Theorem 15, S_{p_n} uniformly converges in a neighbourhood of z_0. But the radius of convergence of the series (5.7) is $R = 1$ and we come to contradiction. □

In conclusion, note that as a particular case one can obtain the corresponding results, referring to the Bessel series, which can be seen in the recently published paper [19].

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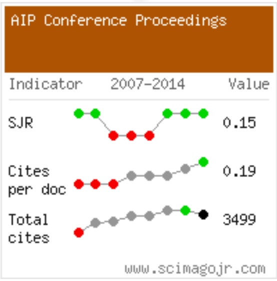

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